Vibrations of Short Beams

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The paper presents a new formulation for transverse oscillations of uniform beams. The governing equations are two simultaneous partial integro-differential equations. From these equations, simpler governing equations to various orders of approximation are deduced. Well-known beam equations correspond to some special cases in the present formulation. Introduction of refined shear coefficient in the Timoshenko's theory seems to increase the discrepancy between theory and experiment, whereas the present formulation reduces this discrepancy. Second-order approximation equations are believed to be adequate for most engineering applications; for more accurate determination of the natural frequency higher-order approximations can be used.

Nomenclature

A	= area of cross section of the beam
b	= width of the beam
b_n,B_{mn},C_{mn}	= cross-sectional constants
d	= depth of the beam
D	= L/d
D^*	= L/nd
E	= Young's modulus
G	= shear modulus
I	= moment of inertia
k_s^2	= $\omega^2\rho L^2/G$
k_v^4	= $\omega^2\rho AL^4/EI$
L	= length of the beam
T	= kinetic energy
U	= strain energy
u,v,w	= displacements along x, y, and z directions, respectively
v_b	= transverse displacement due to bending alone
v_s	= $v - v_b$
$w_n(y)$	= nth warp function
x,y,z	= Cartesian coordinates (x and y are principal axis)
ρ	= mass density
σ_z	= direct stress
σ_{zy}	= shear stress
$\phi_n(z)$	= spanwise variation of nth warp function
ω	= natural frequency

Introduction

THE dynamic behavior of slender structural components may be satisfactorily approximated by considering them as beams. In the past, it was the usual engineering practice to neglect the secondary effects, such as transverse shear and longitudinal inertia, in calculating the natural frequencies of transverse oscillations of beams. This may be justified to some extent for slender beams, at best for few modes of oscillations because, in this case, the influence of secondary effects is small. In short beams, particularly for higher modes, the secondary effects become important and so the elementary beam theory becomes inadequate. Timoshenko[1] extended the domain of validity of beam theory by incorporating the transverse shear and rotary inertia effects into the beam theory. This classical development led to more accurate determination of the frequency spectrum.[2]

Experimental investigations[2,3] have showed that the experimental frequencies are lower than the frequencies obtained from refined beam theories; the discrepancies between theory and experiment are rather small. Nevertheless, an examination of the Timoshenko theory in the light of the work on the shear constant brings out an interesting fact. Recent work,[4] on the shear constant involved in the Timoshenko equation, yielded values to the shear constant as high as 0.870 instead of the original value of $\frac{2}{3}$ proposed by Timoshenko in the case of rectangular cross section. An increase in the value of the shear constant results in an increase in the value of the natural frequency from Timoshenko equation and so it increases the difference between theory and experiment. This increase in the discrepancy (between theory and experiment) is rather small in the case of slender solid beams, but this discrepancy can be large in the case of built-up beams and in the problems of dynamic response and stress waves. Hence, by merely refining the value of the shear constant in the equation proposed by Timoshenko, it is not possible to improve the correlation between theory and experiment.

Within the frame work of the theory of elasticity, starting from three-dimensional equations of equilibrium, one can deduce simpler theories such as beam and plate theories by introducing appropriate simplifying assumptions; the domain of significance of the equations thus, derived depends on the type of assumptions introduced. Medick[5] deduced one-dimensional theories of wave propagation. In Ref. 6, a generalized theory of thin tubes and the deduction of simpler governing equations from the rigorous formulation are presented.

In this paper, we have re-examined the formulation of beam equations. Starting with representative physical assumptions such as zero transverse direct strains and complete freedom to axial displacement, the equations governing the transverse vibrations of beams have been formulated rigorously. Introducing a suitable expression for axial displacement distribution and using the Kantorovich form of Rayleigh-Ritz procedure simpler equations to various orders of approximation are deduced. The well-known elementary beam equation and the Timoshenko theory correspond to some special cases in this formulation.

Consistent with the experimental trends, the present proposal yields results lower than those obtained from the Timoshenko equation. Hence this mathematical model may be considered as a suitable representation for the dynamic behavior of short beams.

Formulation

The main assumption in this paper is that the direct stresses in the plane of the cross section, σ_x and σ_y are zero, and that the Poisson's ratio effect is ignored. The stresses, strains and displacements are assumed to be constant along x direction when the beam bends in the plane of YOZ (see Fig. 1). Further, the cross-sectional shape is assumed to remain undistorted during bending. As x and y are principal

Received January 2, 1969; revision received June 27, 1969.

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axes, there is no coupling between bending about x and y directions.

With these assumptions, we proceed to formulate the equations governing transverse vibrations in the plane of YOZ. The only nonzero stresses, in this case, σ_z and σ_{zy} are given by

$$\sigma_z = E(\partial w/\partial z) \quad (1)$$

$$\sigma_{zy} = G[(\partial w/\partial y) + (dv/dz)] \quad (2)$$

When the beam is executing sinusoidal oscillations with a circular frequency ω, the expression for the maximum kinetic energy of the beam may be written as

$$T = \frac{\rho\omega^2}{2}\int_0^L \int_{-d/2}^{d/2} (v^2 + w^2) bdzdy \quad (3)$$

The maximum strain energy in the beam is

$$U = \frac{1}{2}\int_0^L \int_{-d/2}^{d/2} \left\{E\left(\frac{\partial w}{\partial z}\right)^2 + G\left(\frac{\partial w}{\partial y} + \frac{dv}{dz}\right)^2\right\} bdzdy \quad (4)$$

From the principle of minimum total potential energy, the state of dynamical equilibrium is defined by the condition

$$\delta(U - T) = 0 \quad (5)$$

which yields the governing equations of dynamical equilibrium as

$$GA\frac{d^2v}{dz^2} + \rho\omega^2 Av = -\frac{d}{dz}\int_{-d/2}^{d/2} \frac{\partial w}{\partial y} bdy \quad (6)$$

and

$$E(\partial^2 w/\partial z^2) + G(\partial^2 w/\partial y^2) + \omega^2 \rho w = 0 \quad (7)$$

The boundary condition statement at ends ($z = 0$ or L) can be

$$\text{either } v = 0 \text{ or } \int_{-d/2}^{d/2} G\left(\frac{\partial w}{\partial y} + \frac{dv}{dz}\right) bdy = 0 \quad (8)$$

and

$$\text{either } w = 0 \text{ or } E(\partial w/\partial z) = 0 \quad (9)$$

and at free edges ($y = \pm d/2$),

$$\text{either } w = 0 \text{ or } G\left(\frac{\partial w}{\partial y} + \frac{dv}{dz}\right) = 0 \quad (10)$$

Clearly, from Eq. (6), there is no coupling between the motions in y and z directions if w is symmetric about x axis (even function of y). The governing equations, in this case, reduce to

$$G(d^2v/dz^2) + \omega^2 \rho v = 0 \quad (11)$$

At the ends of the beam,

$$\text{either } v = 0 \text{ or } dv/dz = 0 \quad (12)$$

The equation governing w motion is

$$E(\partial^2 w/\partial z^2) + G(\partial^2 w/\partial y^2) + \rho\omega^2 w = 0 \quad (13)$$

with boundary-condition statements as

$$\begin{aligned} &\text{either } w = 0 \text{ or } \partial w/\partial z = 0 \text{ at } z = 0 \text{ or } L \\ &\text{either } w = 0 \text{ or } \partial w/\partial y = 0 \text{ at } y = \pm d/2 \end{aligned} \quad (14)$$

Now, we shall proceed to examine vibrations in which w displacement is antisymmetric about x axis (odd function of y). One can write the expression for w as

$$w(z,y) = -y\frac{dv}{dz} - \sum_{n=2}^{N} w_n(y)\phi_n(z) \quad (15)$$

The first term in the preceding expression is the one associated

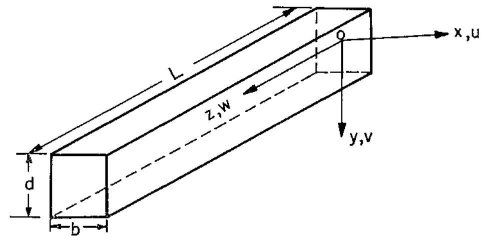

Fig. 1 Short beam.

with elementary beam theory. The expression includes all odd functions in y if w_n is obtained by integrating the previous term w_{n-1} twice as

$$w_n = -\iint w_{n-1} dydy \quad (n = 3,4,5 \ldots N) \quad (16)$$

and

$$w_2 = -\iint ydydy \quad (17)$$

In this scheme, we have two constants of integration associated with each w_n which we utilize to satisfy the free edge zero shear strain condition and to retain the antisymmetric nature of each function in y.†

Using Eq. (15) instead of w and using the Kantorovich form of Rayleigh-Ritz procedure, one can reduce the governing equations to

$$E\left[I\left(\frac{d^4v}{dz^4} + \frac{\rho\omega^2}{E}\frac{d^2v}{dz^2}\right) + \sum_{n=2}^{N} b_n\left(\frac{d^3\phi_n}{dz^3} + \frac{\rho\omega^2}{E}\frac{d\phi_n}{dz}\right)\right] - \omega^2\rho Av = 0 \quad (18)$$

and

$$E\left[b_m\left(\frac{d^3v}{dz^3} + \frac{\rho\omega^2}{E}\frac{dv}{dz}\right) + \sum_{n=2}^{N} B_{m,n}\left(\frac{d^2\phi_n}{dz^2} + \frac{\rho\omega^2}{E}\phi_n\right)\right] + G\sum_{n=2}^{N} C_{m,n}\phi_n = 0 \; (m = 2,3 \ldots N) \quad (19)$$

where

$$I = \int_{-d/2}^{d/2} y^2 bdy, \; A = \int_{-d/2}^{d/2} bdy$$

$$b_m = \int_{-d/2}^{d/2} w_m ybdy, \; B_{m,n} = \int_{-d/2}^{d/2} w_m w_n bdy \quad (20)$$

$$C_{m,n} = \int_{-d/2}^{d/2} \frac{d^2 w_n}{dy^2} w_m bdy \quad (m,n = 2,3 \ldots N)$$

The boundary conditions at the ends can be

$$\text{either } v = 0 \text{ or } E\left[I\left(\frac{d^3v}{dz^3} + \frac{\rho\omega^2}{E}\frac{dv}{dz}\right) + \sum_{n=2}^{N} b_n\left(\frac{d^2\phi_n}{dz^2} + \frac{\rho\omega^2}{E}\phi_n\right)\right] = 0$$

$$\text{either } \frac{dv}{dz} = 0 \text{ or } E\left[I\frac{d^2v}{dz^2} + \sum_{n=2}^{N} b_n\frac{d\phi_n}{dz}\right] = 0 \quad (21)$$

$$\text{either } \phi_m = 0 \text{ or } E\left[b_m\frac{d^2v}{dz^2} + \sum_{n=2}^{N} B_{m,n}\frac{d\phi_n}{dz}\right] = 0$$

$$(m = 2,3 \ldots N)$$

At this stage, we note that two types of vibrations are possible in y direction. One involving no w displacements and governed by Eq. (11) and the other involving w displacements antisymmetric about x axis and governed by Eqs. (18) and (19). The first type of transverse displacement, denoted by

† Development of such function for thin tubes has been discussed in great detail in Ref. 6.

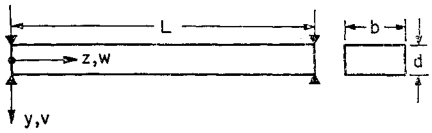

Fig. 2 Simply supported rectangular beam.

v_s involving no w displacement is because of the shearing of the beam whereas the second type, denoted by v_b, which is associated with antisymmetric w displacement, is due to bending of the beam.

Replacing the inertia loads by static loads in Eqs. (11), (18), and (19), one obtains the equations governing the structural behaviour of beams under static loads.[7] An examination of the equations thus obtained brings out that it is possible to have a deformation v_s without giving rise to w or v_b displacement and vice versa and hence there is no static coupling between v_b and v_s. Obviously, since v_b and v_s are in the same direction, the acceleration due to one of them $v_b(v_s)$ causes acceleration to the other $v_s(v_b)$ due to the action of the inertia loads; hence these two are dynamically coupled.

The absence of static coupling between v_b and v_s is probably a limitation of this approach and may be overcome by relaxing some of the assumptions. Nevertheless, the simple fact, that the well known beam equations correspond to some special cases of this formulation, brings out the adequacy of this treatment for beam problems.

The equations governing the coupled motion in y and z directions become

$$G\frac{d^2v_s}{dz^2} + \omega^2\rho(v_s + v_b) = 0$$

$$E\left[I\left(\frac{d^4v_b}{dz^4} + \frac{\omega^2\rho}{E}\frac{d^2v_b}{dz^2}\right) + \sum_{n=2}^{N} b_n\left(\frac{d^3\phi_n}{dz^3} + \frac{\omega^2\rho}{E}\frac{d\phi_n}{dz}\right)\right] - \omega^2\rho A(v_s + v_b) = 0 \quad (22)$$

$$E\left[b_m\left(\frac{d^3v_b}{dz^3} + \frac{\omega^2\rho}{E}\frac{dv_b}{dz}\right) + \sum_{n=2}^{N} B_{m,n}\left(\frac{d^2\phi_n}{dz^2} + \frac{\omega^2\rho}{E}\phi_n\right)\right] + G\sum_{n=2}^{N} C_{m,n}\phi_n = 0 \qquad (m = 2,3\ldots N)$$

The boundary conditions at each end are

$$\text{either } v_s = 0 \text{ or } GA\,dv_s/dz = 0$$

$$\text{either } v_b = 0 \text{ or } E\left[I\left(\frac{d^3v_b}{dz^3} + \frac{\rho\omega^2}{E}\frac{dv_b}{dz}\right) + \sum_{n=2}^{N} b_n\left(\frac{d^2\phi_n}{dz^2} + \frac{\rho\omega^2}{E}\phi_n\right)\right] = 0$$

$$\text{either } \frac{dv_b}{dz} = 0 \text{ or } E\left[I\frac{d^2v_b}{dz^2} + \sum_{n=2}^{N} b_n\frac{d\phi_n}{dz}\right] = 0 \quad (23)$$

and

$$\text{either } \phi_m = 0 \text{ or } E\left[b_m\frac{d^2v_b}{dz^2} + \sum_{n=2}^{N} B_{m,n}\frac{d\phi_n}{dz}\right] = 0 \qquad (m = 2,3\ldots N)$$

Thus, we see that, in a short beam, there are two distinct modes of oscillation. One of them involves w only and is governed by Eq. (13). This is analogous to the two-dimensional membrane equation. The other is a coupled motion in y and z directions and is governed by Eqs. (22). As the primary object of this paper is to develop beam theory for more accurate estimation of natural frequencies of transverse vibrations, hereafter, we concentrate on the solutions of Eqs. (22) only.

Special Cases

When the beam is very long, the shear strains become ignorable and w displacement is practically linear across depth; hence v_s and ϕ_n's can be neglected. With this simplification, Eqs. (22) reduce to

$$EI\frac{d^4v_b}{dz^4} + I\rho\omega^2\frac{d^2v_b}{dz^2} - \rho\omega^2Av_b = 0 \quad (24)$$

which is same as the well-known equation, governing the transverse oscillations of slender beams (when the rotary inertia is included).

But at the supports and near the nodal points the influence of transverse shear strain becomes important. For higher modes for which the wave length is relatively small, this effect assumes even greater importance. Then, it is essential to retain (apart from v_b) v_s, also the governing equations, in this case, are

$$G(d^2v_s/dz^2) + \rho\omega^2(v_b + v_s) = 0$$

and (25)

$$EI\left(\frac{d^4v_b}{dz^4} + \frac{\rho\omega^2}{E}\frac{d^2v_b}{dz^2}\right) - \rho\omega^2A(v_b + v_s) = 0$$

these will be referred hereafter as first-order approximation equations.

One can easily verify that Eqs. (25) are the same as Timoshenko equations; but the shear constant introduced by Timoshenko takes a value of unity, irrespective of the shape of the cross section. Obviously, it must be so in this case, because shear strain is associated with v_s only and it is constant across the cross section.

Further, for more accurate estimation of frequency spectrum, one can retain one term in the expansion for w. The governing equations become

$$G(d^2v_s/dz^2) + \omega^2\rho(v_s + v_b) = 0$$

$$E\left[I\left\{\frac{d^4v_b}{dz^4} + \frac{\omega^2\rho}{E}\frac{d^2v_b}{dz^2}\right\} + b^2\left\{\frac{d^3\phi_2}{dz^3} + \frac{\omega^2\rho}{E}\frac{d\phi_2}{dz}\right\}\right] - \omega^2\rho A(v_s + v_b) = 0 \quad (26)$$

$$E\left[b_2\left\{\frac{d^3v_b}{dz^3} + \frac{\omega^2\rho}{E}\frac{dv_b}{dz}\right\} + B_{2,2}\left\{\frac{d^2\phi_2}{dz^2} + \frac{\omega^2\rho}{E}\phi_2\right\}\right] + GC_{2,2}\phi_2 = 0$$

These will be referred to henceforth, as second-order approximation equations. It is believed that Eqs. (26) are sufficient for most practical applications concerned with transverse oscillations of short beams. Nevertheless, for more accurate determination of the frequency spectrum, one may consider higher approximations.

Cross-Sectional Constants

From Eqs. (22), one may easily notice that the present proposal involves many more cross-sectional constants other than the well known I and A. The determination of these cross-sectional constants, most often, is direct and simple. We illustrate the derivation of these constants by obtaining the constants involved in Eqs. (26) in the case of a rectangular beam. The first step in the derivation of these constants is the determination of the warp functions. Noticing w_2 is an odd function of y and that

$$dw_2/dy = 0 \text{ at } y = \pm d/2 \quad (27)$$

one can obtain w_2 from Eq. (16) as

$$w_2 = (y/24)(3d^2 - 4y^2) \quad (28)$$

Table 1 Frequency parameter, k_s^2 obtained by solving Eqs. (25) for simply supported beams

	$k_s^2 = \omega^2 \rho L^2/G$	
$D^* = L/nd$	First set	Second set
2	3.19352197	80.830534
4	1.13772440	226.88633
6	0.55219174	467.47187
8	0.32118034	803.70288
10	0.20887947	1235.8152
15	0.09434509	2735.9297

Table 2 Frequency parameter, k_s^2 obtained by solving Eqs. (26) for simply supported beams

	$k_s^2 = \omega^2 \rho L^2/G$		
$D^* = L/nd$	First set	Second set	Third set
2	2.5058121	46.894884	7452.7797
4	1.00674307	115.82834	29705.351
6	0.51884483	225.43078	66796.242
8	0.30939981	337.970336	118723.93
10	0.20378623	573.810791	185488.17
15	0.09328742	1253.3916	417308.83

Using Eq. (28) in Eqs. (20), we get

$$I = (1/12)bd^3$$
$$A = bd$$
$$b_2 = -C_{2,2} = bd^5/120$$
$$B_{2,2} = (17/20160)bd^7 \quad (29)$$

Simply Supported Rectangular Beam

In this section, we present the solutions of a simply-supported rectangular beam (see Fig. 2) by using governing equations to various orders of approximation. In the next section, we discuss the limitations of various beam theories, based on the numerical results, from the formulas presented in this section. The simply-supported end is mathematically described as

$$v = \sigma_z = 0 \quad (30)$$

Using Eq. (24), one can easily obtain the well-known expression for the frequency parameter as

$$k_s^2 = n^4\pi^4 EI/G(n^2\pi^2 I + AL^2) \quad (n = 1,2,3\ldots) \quad (31)$$

and the mode shape is

$$v_b = A\sin(n\pi z)$$

and

$$v_s = 0 \quad (31a)$$

Neglecting rotatory inertia, one obtains from Eq. (31)

$$k_v^4 = n^4\pi^4 \ (n = 1,2,3\ldots) \quad (32)$$

where

$$k_v^4 = \omega^2\rho AL^4/EI = k_s^2(GAL^2/EI)$$

Using Eqs. (25), one obtains the values of the frequency parameter k_s^2 as the solution of the quadratic equation

$$\xi^2 - b\xi + c = 0 \quad (33)$$

where

$$b = n^2\pi^2(k^2+1) + \mu^2$$
$$c = k^2 n^4 \pi^4 \quad (n = 1,2,3\ldots)$$

and

$$k^2 = E/G, \ \mu^2 = 12D^2 \quad (34)$$

The corresponding mode shape in this case is

$$v_b = B\sin(n\pi z) \quad (n = 1,2,3\ldots)$$

$$v_s = \frac{k_s^2}{n^2\pi^2 - k_s^2} B\sin(n\pi z) \quad (n = 1,2,3\ldots) \quad (35)$$

Solving Eqs. (26), the value of the frequency parameter k_s^2 is obtained as the solution of the cubic equation

$$p_1\xi^3 + p_2\xi^2 + p_3\xi + p_4 = 0 \quad (36)$$

where

$$p_1 = \nu_1^2 - \nu_2^2$$
$$p_2 = X_2 + \nu_2^2 b - \nu_1^2 n^2\pi^2(1+2k^2)$$
$$p_3 = k^2\nu_1^2 n^4\pi^4(2+k^2) - X_2 b - \nu_2^2 c$$
$$p_4 = c(-\nu_1^2 k^2 n^2\pi^2 + X_2)$$

and

$$X_2 = k^2\nu_2^2 n^2\pi^2 + 1 \quad (n = 1,2,3\ldots)$$
$$\nu_1^2 = 1/10D^2; \quad \nu_2^2 = 17/168D^2 \quad (37)$$

Results and Discussion

For a simply-supported beam, its higher harmonic corresponds to the fundamental of another simply-supported beam of shorter span; the nth frequency of simply-supported beam of span L is equal to the fundamental of another such beam with span L/n. So in this section, for the sake of simplicity, we discuss the variation of the frequency parameters with the parameter D^*.

Equations (25) and (26) provide us with more frequencies than are accounted in the elementary beam theory. Table 1 shows the values of the frequency parameter obtained by solving Eqs. (25) for various values of D^*.

Using Eq. (35), it can easily be deduced, that in the first set of frequencies v_s is ignorably small compared to v_b, whereas for the second set they are of comparable magnitude. As the second set is so widely separated from the first, it is of little practical importance. Table 2 presents the frequency spectrum exhibited by Eqs. (26).

Again, a study of the nature of mode shape associated with each set of the frequencies brings out that second and third sets involve primarily axial motion whereas the first set involve primarily transverse motion. Obviously second and third set are of little practical importance.

Table 3, shows the comparison of frequencies associated with primarily transverse oscillations of simply-supported beams. In this table, for the sake of simplicity, we use the frequency parameter k_v^4, whose value by elementary beam theory is π^4 and does not depend on D^*.

Table 3 Comparison of k_v^4 by various equations $k_v^4 = \omega^2\rho AL^4/EI$

	k_v^4 obtained by solving	
D^*	Eqs. (25)	Eqs. (26)
2	57.958	45.388
4	82.431	72.941
6	90.017	84.580
8	93.081	89.668
10	94.587	92.278
15	96.125	95.047
k_v^4 by elementary theory $= \pi^4 = 97.408$		

Table 4 Comparison of frequency parameter $k_v{}^{*4}$

D^*	From Timoshenko theory with shear coefficient equal to 2/3	0.870	By Eqs. (25)	By Eqs. (26)
2	0.517	0.569	0.595	0.465
4	0.770	0.834	0.846	0.748
6	0.899	0.915	0.924	0.868
10	0.958	0.968	0.971	0.947
20	0.989	0.991	0.992	0.986
50	0.997	0.998	0.988	0.997

Comparison with Timoshenko Theory

As mentioned earlier, Timoshenko theory corresponds to Eqs. (25), but with unit value for Timoshenko shear constant. Originally Timoshenko gave a value of $\frac{2}{3}$ to this coefficient. Later workers, using more sophisticated approaches, proposed different values to this constant. Recently Cowper[4] proposed an expression for the Timoshenko shear constant. In Table 4, we compare the values of the frequency parameters obtained from Timoshenko equations with two typical values for shear coefficient with the results from Eqs. (25) and (26).

In Table 4, we use the frequency parameter $k_v{}^{*4}$ defined as

$$k_v{}^{*4} = (\omega^2 \rho A L^4/EI)/n^4\pi^4$$

which takes a value of unity by elementary beam theory.

We may mention here that the natural frequencies, associated with primarily transverse oscillation, by the present theory (second-order approximation) are within 1% of the Timoshenko theory for $D^* > 15$.

It can be seen from Table 4, that as the value of the shear constant increases, there is an increase in the value of the frequency parameter obtained by the use of the Timoshenko's equation. Experimental results reported in the literature[2,3] have indicated that the experimental frequencies are lower than those predicted by refined beam theories. The second-order approximation, which is a more sophisticated mathematical model than the Timoshenko theory, also yields lower values to the frequency parameter than those obtained from Timoshenko's equation. Hence, a refinement of only the shear coefficient in Timoshenko equation leads to an increase in the discrepancy between theory and experiment unless the basic mathematical model is also suitably modified. The present approach yields results consistent with experimental trends and so it can be considered as a suitable mathematical model for the dynamical behavior of short beams.

Conclusions

In this paper, we proposed a new formulation for transverse oscillations of uniform beams. The governing equations are two simultaneous partial integro-differential equations. Using physical features of the problem, simpler governing equations to various orders of approximation are deduced. The well-known beam equations correspond to some special cases in the present formulation.

The present proposal brings out more frequencies than are accounted in the elementary beam theory. These extraneous frequencies, being widely separated from the frequencies associated with primarily transverse vibrations, are of little practical importance.

By merely refining the shear constant in Timoshenko equation, it is not possible to reduce the differences between theory and experiment as the experimental frequencies are lower than those predicted by Timoshenko equation and as the refinement of the shear constant in Timoshenko equation increases the theoretical frequency. Consistent with the experimental trends the present proposal yields frequencies lower than those obtained from Timoshenko equation. As such, the present proposal can be considered as a more suitable representation of the dynamic behavior of short beams.

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